Review Problems For Stat 311

1. Fred is planning to go to dinner each night of this week (Mon-Fri), with each dinner being in one of 10 restaurants. How many possibilities are there if Fred will not go to the same restaurant twice in a raw (or more)? That is, he can go to the same restaurant several times, but not consecutively.
2. 20 people in a chess club divide in pairs and start playing. How many playing arrangements are possible if it matters which player is assigned to play the white pieces?
3. A taxi driver in Manhattan must navigate a 4 X 5 city grid starting from (0, 0) and ending in (4, 5). The driver must also pick up an additional passenger located at (2, 2). How many ways are there to get to the final destination if going back isn't allowed? (По несчастью или к счастью, Истина проста: Никогда не возвращайся В прежние места.)
4. There are 15 chocolate bars and 10 children. In how many ways can the chocolate bars be distributed to the children if

   (a) The chocolate bars are identical

   (b) The chocolate bars are identical and each child must receive at least one.

   (c) The chocolate bars are all distinct.

   (d) The chocolate bars are all distinct and each child must receive at least one.

5. 3 residents enter an elevator in an apartment complex at the ground floor. They each press a button 1-10 (each resident presses the button that corresponds to his or her floor. The same button may be pressed several times). If each button is equally likely to be pressed, what is the probability that 3 consecutive buttons are pressed?
6. 2 standard decks of cards are mixed into one (Thus there are 104 cards). 5 cards are randomly withdrawn. What is the probability of two pairs? (a, a, b, b, c)
7. King Arthur, Sir Lancelot, and another 10 knights are seating at a round table, with their seating arrangement having been randomly assigned. What is the probability that Arthur and Lancelot are sitting next to each other? Compute this in two ways

   (a) Using a sample space of size 12!, where an outcome carries full seating arrangement details.

   (b) Sample space focusing solely on king Arthur and sir Lancelot.

בתוך ים סוער
רק שנינו נישאר
כי לאהבה אין מדינה

8. There are k distinguishable balls and n distinguishable boxes. The balls are randomly placed in the boxes, with all n^k possibilities equally likely. Problems in this setting are called *occupancy problems*, and (apparently) are at the core of many widely used algorithms in computer science.

(a) Find the expected number of empty boxes.

(b) Find the probability that at least one box is empty.

(c) Let n = 1000, k = 5806. Find a good approximation as a decimal for the probability that at least one box is empty (Hint: Poisson)

9. A survey is being conducted in a city of 1000, 000 residents. For that purpose, a random sample of size 1000 is chosen. The survey is conducted by choosing people one at a time, with replacement and with equal probabilities. Estimate the probability that at least one person is chosen more than once. (Hint: Poisson)
10. n well shuffled cards (numbered 1-n) are overturned one at a time. The player wins if at least one card, say card k, is the kth card overturned. What is the probability of winning? What does this probability tend to when n is very large?
11. Every day, a particular website is independently visited by any one of 10 million people. The probability that one of them decides to visit the website is $p = 2 \times 10^{-7}$. Approximate the probability of at least 3 visits.

12. Use combinatorial arguments to prove

(a) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

(b) $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

(c) $\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}$

(d) $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$

(e) $\sum_{k=1}^{n} \binom{k}{2} \binom{n-k+2}{2} = \binom{n+3}{5}$
(Hint: Consider the middle number in a subset of {1, 2,..., n+3} of size 5.)

13. A package of Haribo gummy bears contains anywhere between 30 and 50 bears. The bears come in 5 flavors: Pineapple (clear), raspberry (red), orange (orange), strawberry (green), and lemon (yellow). How many possibilities of filling the bag if every type of bear must be present in the bag? (Hint: Use identity (d) from the previous exercise.)
14. Every given minute an ameba is equally likely to do one of 3 things: (1) Die (2) Divide into two (3) Stay the same. If the ameba divides, each of the two cells will continue the above cycle

independently. What is the probability that the ameba colony generated from a single cell will eventually die out?

15. (Student Capture-Recapture) In a certain probability course of 30 CS majors, 3 students are caught cheating on the midterm. They are tagged and released with a warning back into the wild. All the 30 students practice the Spartan code: "You're a thief only if you were caught stealing". Those who are caught, never learn their lesson. After the final, a random sample of 10 students is drawn and analyzed for signs of cheating.

    (a) What's the probability that at least one of the 3 midterm cheaters is in the sample?

    (b) What is the expected number of midterm cheaters that is captured within this sample?

16. Each serial box is equally likely to contain any one of n superheroes. What is the expected number of boxes the parents of the child will have to buy for the kid to have collected all superhero types? (If you don't like to solve for abstract n, assume n = 5, although we all know that $n \to \infty$)
17. (Putnam Problem) A permutation $(a_1, a_2, \dots, a_n)$ has a local maximum at j if $a_j > a_{j-1}$ and $a_j > a_{j+1}$ for $2 \leq j \leq n-1$. A local maximum at a_1 means $a_1 > a_2$ and a local maximum at a_n means $a_n > a_{n-1}$. For example, the permutation (4, 2, 5, 3, 6, 1) has 3 local maxima (at 4, 5, and 6). What is the average number of local maxima of a random permutation of 1, 2, ..., n?
18. (This one is hard!) An urn contains m white balls and n black balls. The balls are randomly sampled one by one until a total of r white balls have been gathered. What is the probability that k black balls are collected? What is the expected number of black balls in this sample?
19. A gram of radio-active material emits, on average, 3 alpha particles per minute. What is the probability that 5 alpha particles will be released in the next 2 minutes? What is the probability of 12 alpha particles in the next 3 minutes?
20. There are 10 equally spaced points around the circle. At 9 points, there are sheep, and at one point, there is a wolf. At each time step, the wolf moves clockwise with probability p or counterclockwise with probability 1-p by 1 point. If there is a sheep at that point, the wolf eats it. The sheep don't move. What is the probability that the sheep who is initially behind the wolf is the last one remaining?
21. The number of people that enter a drug store in a given hour is a Poisson random variable with parameter $\lambda = 10$. Compute the conditional probability that at most 3 men entered the drug store, given that 10 women entered in that hour. What assumptions have you made?
22. Ebenezer Scrooge is visited by the ghosts of Christmas Past, of Christmas Present, and of Christmas Yet to Come. If these ghosts decide to visit him randomly between 12:00 and 1:00 am, what is the probability that they will all arrive in the right order?
23. An infinite sequence of numbers in the interval [0,1] is randomly generated. What is the probability that this sequence is convergent? If you took analysis, what is the probability this sequence is dense in [0, 1]? Would your answer be any different if another (non-uniform) probability density function is used to generate this sequence?

24. A needle of length L is randomly dropped on a ruled sheet of paper. If the width of a ruled strip is H (H > L), what is the probability that this needle intersects one of the ruled lines?